Exchange Forces of Composite Particles in Quantum Field Theory

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Quantum fields can be characterized by state functionals and corresponding functional equations. Within this functional representation exchange forces of composite particles are discussed for the case of composite bosons which are bound states of two constituent fermions. The dynamics of these bosons is formulated by means of a weak mapping theorem which establishes a map between the functional equations for the composite boson quantum field and the constituent original fermion quantum field. Evaluation of this theorem leads to expressions which can be identified as quantum field theoretic "direct" forces and exchange forces for or between composite particles. By some theorems the exchange forces are evaluated and an estimate for them is given. The expressions for the direct forces correspond to those which were already derived in previous papers to discuss composite particle dynamics.

Key words: PACS 11.10: Field theory; PACS 12.10: Unified field theories and models; PACS 12.35: Composite models of particles.

Introduction

One of the most outstanding problems in any atomistic theory of matter is to derive the formation and reactions of composite particles by means of the dynamics of their elementary atomistic constituents. In quantum mechanics this led to the discovery of the fermion exchange forces between (and inside) composite particles which have no classical counterpart but nevertheless play an essential role in the formation of matter. Thus if on a more advanced quantum theoretical level matter is described by quantum field theory one should rediscover these exchange forces in field theoretic composite particle formalism. Although in the past numerous efforts were made to develop such a nonrelativistic and relativistic composite particle formalism in quantum field theory, so far no satisfactory and systematic answers have been obtained for the solution of this problem. This deficiency is essentially related to the fact that so far field operator products were erroneously identified with the description of composite particles. Apart from technical difficulties with path integrals etc. it was shown [1] that this leads, for instance, to a formal bosonization of fermionic theories which is not uniquely related to the description of physical reactions by bound states, i.e., definite composite particles. In consequence of this, with operator product techniques it is also impossible to derive exchange forces correctly.

The use of field operator products for the derivation of effective actions can be referred to as strong mapping. In order to avoid the drawbacks of strong mappings an alternative program has been started by one of the authors which can be referred to as weak mapping of quantum fields [2].

Quantum field theories can be characterized by functional equations. In contrast to the strong mapping by means of operator products, in the weak mapping procedure *functional equations* are mapped on to *effective functional equations* in order to obtain an effective composite particle dynamics. These weak mappings respect the algebraic properties of quantum fields and are free of the difficulties and insufficiencies mentioned above, provided the basic quantum field theories are sufficiently regularized.

The weak mapping has been exemplified for basic spinor fields with four-fermion interactions and nonperturbative Pauli-Villars regularization. In preceding papers this mapping has been studied without taking into account exchange terms [2]. In this paper the effects of exchange terms are included, leading thus to a complete mapping procedure between quantum field theoretical functional equations.

In order to make the deduction as transparent as possible we demonstrate this mapping again for the spinor field theory already used. Furthermore, we

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confine ourselves to the special mapping of spinor field functional equations onto boson field functional equations.

A weak mapping theorem between such functional equations was already derived in [1]. In contrast to the technique used in Sect. 3, in the proof of this theorem in [1] a map between functional spaces was considered and in consequence a separation of "ordinary" and exchange forces is not evident. Thus the theorems in Sect. 4 which are based on the weak mapping theorem of Sect. 3 for the first time allow this separation of forces. As both weak mapping theorems are concerned with the same problem they have to be equivalent. A proof of the equivalence will be given elsewhere.

For abbreviation, in the following a condensed notation is used:

$$\partial_\mu := \frac{\partial}{\partial x_\mu} \quad \text{for greek small letter index,}$$

$$\partial_I := \frac{\delta}{\delta j_I} \quad \text{for latin capital letter index,}$$

$$\partial_k := \frac{\delta}{\delta b_k} \quad \text{for latin small letter index,}$$

$\{A_1 \dots A_n\}$:= antisymmetrization of $A_1 \dots A_n$ if not otherwise stated;
in contrast $\{a_n\}$ means the set $a_1 \dots a_\infty$ etc.

The summation convention is used throughout the paper.

1. Fundamentals of the Model

The general spinor field model which is assumed to be the basis of the theory is defined by the field equations ($\hbar = c = 1$)

$$\delta_{AB}(-i\gamma^\varrho \partial_\varrho + m)^{\text{reg}}_{\alpha\beta}\, \psi_{\beta B}(x) = g\, V^{ABCD}_{\alpha\beta\gamma\delta}\, \psi_{\beta B}(x)\, \bar\psi_{\gamma C}(x)\, \psi_{\delta D}(x), \tag{1.1}$$

where the indices $\alpha, \beta, \dots$ and $A, B, \dots$ describe spin and isospin, respectively. The regularization is defined by

$$(-i\gamma^\varrho \partial_\varrho + m)^{\text{reg}} := (-i\gamma^\varrho \partial_\varrho + m)(-i\gamma^\mu \partial_\mu + m_1)(-i\gamma^\lambda \partial_\lambda + m_2). \tag{1.2}$$

Due to the mass terms in (1.1) or (1.2), the corresponding spinor field has to be a Dirac spinor-isospinor. Furthermore, the vertex operator is assumed to have the form

$$V^{ABCD}_{\alpha\beta\gamma\delta} = \frac{1}{2} \sum_{h=1}^{2} \left(v^{h\,AB}_{\alpha\beta}\, v^{h\,CD}_{\gamma\delta} - v^{h\,AD}_{\alpha\delta}\, v^{h\,CB}_{\gamma\beta} \right) \tag{1.3}$$

with

$$v^{h\,AB}_{\alpha\beta} := \hat v^h_{\alpha\beta}\, \delta_{AB}, \quad h = 1, 2,$$
$$\hat v^1_{\alpha\beta} := \delta_{\alpha\beta}; \qquad \hat v^2_{\alpha\beta} := i\gamma^5_{\alpha\beta}. \tag{1.4}$$

The special form of the vertex operator is of no consequence for the general mapping procedure. With respect to its physical meaning we refer to preceding papers [2].

According to the decomposition theorem [3] it can be proved that the set of nonlinear equations $r = 0, 1, 2$

$$\delta_{AB}(-i\gamma^\mu \partial_\mu + m_r)_{\alpha\beta}\, \varphi_{\beta B r}(x) = g\,\lambda_r \sum_{stu} V^{ABCD}_{\alpha\beta\gamma\delta}\, \varphi_{\beta B s}(x)\, \bar\varphi_{\gamma C t}(x)\, \varphi_{\delta D u}(x) \tag{1.5}$$

with $m_0 \equiv m$, is connected with (1.1) by a bi-unique map defined by the compatible relations

$$\psi_{\alpha A}(x) = \varphi_{\alpha A 0}(x) + \varphi_{\alpha A 1}(x) + \varphi_{\alpha A 2}(x),$$
$$\varphi_{\alpha A r}(x) = \lambda_r\, [(-i\gamma^\varrho \partial_\varrho + m_{r+1}) \cdot (-i\gamma^\mu \partial_\mu + m_{r+2})]_{\alpha\beta}\, \delta_{AB}\, \psi_{\beta B}(x), \tag{1.6}$$
$$\lambda_r = (m_r - m_{r+1})^{-1} (m_r - m_{r+2})^{-1}, \quad r = 0, 1, 2 \text{ cyclic}.$$

By formulating the corresponding Lagrangean for (1.5) it is easily seen that (1.5) is the nonperturbative Pauli-Villars regularization of (1.1).

It is furthermore convenient to use the charge conjugated spinor instead of the adjoint spinor, which is defined by

$$\varphi^c_{\nu A j} = C^{-1}_{\nu\nu'}\, \bar\varphi_{\nu' A j}. \tag{1.7}$$

Introducing the superspinors

$$\varphi_{\alpha A j 1} := \varphi_{\alpha A j}; \quad \varphi_{\alpha A j 2} := \varphi^c_{\alpha A j}, \tag{1.8}$$

we can formally combine (1.5) and its charge conjugated equations into one equation:

$$(D^\mu_{Z_1 Z_2}\, \partial_\mu - m_{Z_1 Z_2})\, \varphi_{Z_2}(x) = \sum_h U^h_{Z_1 \{Z_2 Z_3 Z_4\}}\, \varphi_{Z_2}(x)\, \varphi_{Z_3}(x)\, \varphi_{Z_4}(x) \tag{1.9}$$

with $Z := (\alpha, A, i, \Lambda)$ and

α = spinor index ($\alpha = 1, 2, 3, 4$),
A = isospin index ($A = 1, 2$),
i = auxiliary field index ($i = 0, 1, 2$),
Λ = superspinor index ($\Lambda = 1, 2$), (1.10)

where the following definitions are used:

$$D^{\mu}_{Z_1Z_2} := i\,\gamma^{\mu}_{\alpha_1\alpha_2}\,\delta_{A_1A_2}\,\delta_{i_1i_2}\,\delta_{\Lambda_1\Lambda_2}\,,$$

$$m_{Z_1Z_2} := m_{i_1}\,\delta_{\alpha_1\alpha_2}\,\delta_{A_1A_2}\,\delta_{i_1i_2}\,\delta_{\Lambda_1\Lambda_2}\,, \qquad (1.11)$$

$$U^h_{Z_1Z_2Z_3Z_4} := g\,\lambda_{i_1}\,\hat{v}^h_{\alpha_1\alpha_2}\delta_{A_1A_2}\delta_{\Lambda_1\Lambda_2}(\hat{v}^h C)_{\alpha_3\alpha_4}\delta_{A_3A_4}\delta_{\Lambda_3 1}\delta_{\Lambda_4 2}.$$

By using the Lagrangean formulation of (1.5), the canonical quantization procedure leads to the non-vanishing equal time anticommutator

$$[\varphi_Z(\boldsymbol{r},t), \varphi_{Z'}(\boldsymbol{r}',t)]_+ = A_{ZZ'}\,\delta(\boldsymbol{r}-\boldsymbol{r}')\,,$$

$$A_{ZZ'} := \lambda_j\,\delta_{jj'}\,\delta_{AA'}\,\sigma^1_{\Lambda\Lambda'}(C\gamma^0)_{\alpha\alpha'}\,. \qquad (1.12)$$

The unknown representation space $\mathscr{H}$ of the theory is spanned by the eigenstates of a complete set of commutating observables. Following the general principles of quantum field theory, any state $|\alpha\rangle \in \mathscr{H}$ may be characterized by the set of τ-functions (time-ordered)

$$\tau_n(x_1, Z_1, \ldots, x_n, Z_n|a) \qquad (1.13)$$
$$= \langle\Omega|\pi(T\psi_{Z_1}(x_1)\ldots\psi_{Z_n}(x_n))|a\rangle,\ n\in\mathbb{N},\ |a\rangle\in\mathscr{H},$$

where π denotes the representation and $|\Omega\rangle$ the corresponding vacuum state.

The algebraic representation of the system is already fixed by considering the equal time limits in (1.13) to a space-like hypersurface σ. Choosing a special Lorentz frame we may transform σ to the flat hypersurface $t=0$ leading to antisymmetrized operator product matrix elements

$$a_n(\boldsymbol{r}_1, Z_1, \ldots, \boldsymbol{r}_n, Z_n|a)$$
$$= \langle\Omega|\pi(\mathscr{A}\psi_{Z_1}(\boldsymbol{r}_1,0)\ldots\psi_{Z_n}(\boldsymbol{r}_n,0))|a\rangle$$
$$n\in\mathbb{N},\quad |a\rangle\in\mathscr{H},\quad \boldsymbol{r}_i\in\mathbb{R}^3$$

which are the basic quantities of the field theoretical energy representation.

Furthermore, in order to obtain a compact description of the field dynamics it is necessary to introduce generating functional states for the sets (1.13) or (1.14), resp., with Grassmann algebraic sources $j_Z(x)$ or $j_Z(\boldsymbol{r})$, resp., and to transform the field equations (1.9) into corresponding functional equations. Thus in the following we use for the discussion of the sets (1.13) or (1.14), resp., these generating functional states.

2. Functional Energy Representation

As already mentioned, the time-ordered state functionals are redundant with respect to algebraic representation theory. Therefore, for the successful application of weak mappings the corresponding time-ordered functional equations are not suitable. Their redundancy has to be removed from the theory, which is achieved by the transition to antisymmetric state functionals

$$|\mathscr{A}[j,a]\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\int a_n(\boldsymbol{r}_1, Z_1\ldots\boldsymbol{r}_n, Z_n|a)$$
$$\cdot\, j_{Z_1}(\boldsymbol{r}_1)\ldots j_{Z_n}(\boldsymbol{r}_n)|0\rangle\,\mathrm{d}^3r_1\ldots\mathrm{d}^3r_n \qquad (2.1)$$

and their corresponding equations. However, even these state functionals are not suitable for composite particle theory because they contain disconnected parts. For instance, for composite bosons which are bound states of two constituent fermions all disconnected parts due to the Fermion propagator $F_{\alpha\alpha'}(x-x')$ have to be removed from (2.1) in order to obtain a correct effective boson theory. Thus if we restrict ourselves to this case we have to use the normal transform $|\mathscr{F}\rangle := Z_F[j]|\mathscr{A}\rangle$ instead of $|\mathscr{A}\rangle$ itself.

A careful analysis yields the equation for $|\mathscr{F}\rangle$:

$$(E_a - E_0)|\mathscr{F}\rangle = \int j_Z(\boldsymbol{r})\,D^0_{ZZ_1}[D^k_{Z_1Z_2}\nabla_k - m_{Z_1Z_2}]\,\partial_{Z_2}(\boldsymbol{r})\,\mathrm{d}^3r\,|\mathscr{F}\rangle$$
$$+\sum_h\int j_Z(\boldsymbol{r})\,iD^0_{ZZ_1}\,U^h_{Z_1\{Z_2Z_3Z_4\}}:d_{Z_4}(\boldsymbol{r})\,d_{Z_3}(\boldsymbol{r})\,d_{Z_2}(\boldsymbol{r}):\mathrm{d}^3r\,|\mathscr{F}\rangle \qquad (2.2)$$
$$+\tfrac{1}{4}\sum_h\int j_Z(\boldsymbol{r})\,iD^0_{ZZ_1}\,U^h_{Z_1\{Z_2Z_3Z_4\}}\,A_{Z_4Z'}A_{Z_3Z''}\,j_{Z'}(\boldsymbol{r})\,j_{Z''}(\boldsymbol{r})\,d_{Z_2}(\boldsymbol{r})\,\mathrm{d}^3r\,|\mathscr{F}\rangle$$

with

$$d_Z(\boldsymbol{r}) := \partial_Z(\boldsymbol{r}) - \int F^a_{ZZ'}(\boldsymbol{r},\boldsymbol{r}')\,j_{Z'}(\boldsymbol{r}')\,\mathrm{d}^3r'\,, \qquad (2.3)$$

where F^a in (2.3) is the antisymmetric equal time limit of the free field propagator F and A is the anticommutator in (1.12). With the abbreviations

$$K_{I_1I_2} := i\,D^0_{Z_1Z}(D^k_{ZZ_2}\nabla_k - m_{ZZ_2})\,\delta(\boldsymbol{r}_1-\boldsymbol{r}_2), \qquad (2.4)$$

$$W^h_{I_1I_2I_3I_4} := i\,D^0_{Z_1Z}\,U^h_{Z\{Z_2Z_3Z_4\}}$$
$$\cdot\,\delta(\boldsymbol{r}_1-\boldsymbol{r}_2)\,\delta(\boldsymbol{r}_1-\boldsymbol{r}_3)\,\delta(\boldsymbol{r}_1-\boldsymbol{r}_4), \qquad (2.5)$$

$$A_{I_1I_2} = A_{Z_1Z_2}\,\delta(\boldsymbol{r}_1-\boldsymbol{r}_2),$$

(2.2) can be written in the compact form

$$\omega_{a0}|\mathscr{F}[j]\rangle = K_{I_1I_2}\,j_{I_1}\,\partial^{I_2}|\mathscr{F}[j]\rangle$$
$$+\sum_h W^h_{I_1I_2I_3I_4}\{j_{I_1}\,\partial^{I_4}\partial^{I_3}\partial^{I_2} - 3F^a_{I_4I}\,j_{I_1}j_I\,\partial^{I_3}\partial^{I_2}$$
$$+(3F^a_{I_4I}\,F^a_{I_3I'} + \tfrac{1}{4}A_{I_4I}\,A_{I_3I'})\,j_{I_1}j_Ij_{I'}\,\partial^{I_2} \qquad (2.6)$$
$$-(F^a_{I_4I}\,F^a_{I_3I'} + \tfrac{1}{4}A_{I_4I}\,A_{I_3I'})\,F^a_{I_2I''}\,j_{I_1}j_Ij_{I'}j_{I''}\}\,\mathscr{F}[j]\rangle,$$

where $\omega_{a0}=(E_a-E_0)$ is the renormalized energy. Equation (2.6) is the functional energy representation of the spinor field theory. It is the basic equation with respect to composite boson dynamics.

In order to perform weak mappings we have to define the composite particle states. Since composite particles have interactions they cannot be described by exact solutions of (2.6), where all interactions are taken into account within these solutions. Thus composite particles are only a part of the total system and we are forced to look for their mathematical counterpart with respect to (2.6). To obtain composite particle states, we consider by definition only the "diagonal part" of (2.6), which is given by all particle number conserving contributions of this equation [2]:

$$p_0|\mathscr{F}\rangle = j_{I_1} K_{I_1 I_2} \partial_{I_2}|\mathscr{F}\rangle - \sum_h j_{I_1} W^h_{I_1 I_2 I_3 I_4} 3 F^a_{I_4 I'} j_{I'} \partial_{I_3} \partial_{I_2}|\mathscr{F}\rangle. \quad (2.7)$$

The sets of solutions of (2.7) can be assumed to be complete sets in $\mathbb{R}^{3n}$ $(n=1\ldots)$. In particular we denote the set of two particle solutions by $\{|\mathscr{F}_k\rangle = C_k^{I_1 I_2} j_{I_1} j_{I_2},\ k=1\ldots\}$.

Due to the antisymmetry $C_k^{I_1 I_2}=-C_k^{I_2 I_1}$ even the equal time functions $\{C_k^{I_1 I_2}\}$ are redundant, because their domain of definition $\{I_1, I_2\}$ can be subdivided into equivalence classes by means of a general ordering relation $I_1<I_2$. Choosing only one representative, one can remove this redundancy. Thus we have to consider the set $\{C_k^{I_1 I_2}\}$ on the ordered domain $\{I_1, I_2, I_1<I_2\}$. This set contains bound states as well as scattering states for spin zero and spin one configurations. Thus we can assume that it is a complete set of two-fermion states. In spite of the completeness it is in general *not* an orthogonal set, since the eigenvalue equation (2.7) is *not* a Schrödinger equation and the solutions $\{C_k^{I_1 I_2}\}$ have in general not the meaning of quantum mechanical probability amplitudes. About the connection of $\{C_k^{I_1 I_2}\}$ with statistics see [4]. The non-Schrödinger properties of the set $\{C_k^{I_1 I_2}\}$ are of no relevance for the weak mapping theorem. It suffices to introduce a dual set $\{R^k_{I_1 I_2}\}$ which is defined by the orthogonality and completeness relations

$$\sum_{I_1<I_2} R^k_{I_1 I_2} C_{k'}^{I_1 I_2} = \tfrac{1}{2}\delta^k_{k'}, \quad (2.8)$$

$$\sum_k R^k_{I_1 I_2} C_k^{I'_1 I'_2} = \delta^{I'_1}_{I_1}\delta^{I'_2}_{I_2}, \quad I_1<I_2, \quad I'_1<I'_2.$$

It is convenient to rewrite (2.8) for the whole domain. This gives

$$\sum_k R^k_{I_1 I_2} C_k^{I'_1 I'_2} = \tfrac{1}{2}(\delta^{I'_1}_{I_1}\delta^{I'_2}_{I_2} - \delta^{I'_1}_{I_2}\delta^{I'_2}_{I_1}). \quad (2.9)$$

In the same way the orthonormality relation can be rewritten for the whole domain as

$$\sum_{I_1 I_2} R^k_{I_1 I_2} C_{k'}^{I_1 I_2} = \delta^k_{k'}. \quad (2.10)$$

As the initial set $\{C_k^{I_1 I_2}\}$ is explicitly known, the dual set $\{R^k_{I_1 I_2}\}$ can explicitly be constructed.

3. Weak Mapping Theorem

We study functional mappings of (2.6) which we exemplify for the case of a transformation to a boson theory. In order to perform such a mapping we have to observe that the functional spaces are only auxiliary embedding spaces which allow a compactified description of the field dynamics. Thus the mapping will be concerned with the physically relevant matrix elements (1.14) and not with the state functionals (2.1) themselves. Nevertheless, we expect that the result of the mapping procedure can finally be expressed again by functional states and equations. Therefore we postulate that the weak mapping of (2.1) can be formulated by a boson state functional

$$|\mathscr{B}[b]\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \varrho^{k_1\ldots k_n} b_{k_1}\ldots b_{k_n}|0\rangle, \quad (3.1)$$

where the functional embedding space is defined by the relations

$$[\partial_k\, \partial_{k'}]_- = [b_k\, b_{k'}]_- = 0,$$
$$[\partial_k\, b_{k'}]_- = \delta_{kk'}; \quad \partial_k|0\rangle = 0 \quad (3.2)$$

with ∂_k the dual of b_k, i.e. $\partial_k = b_k^+$. This boson functional space is completely independent of the fermionic functional space introduced in (2.1). Rather the relation between (3.1) and (2.1) is defined by the expression

$$a^{I_1\ldots I_{2n}} = (-1)^n C_{k_1}^{\{I_1 I_2}\ldots C_{k_n}^{I_{2n-1} I_{2n}\}} \varrho^{k_1\ldots k_n}, \quad (3.3)$$

where $\{\ \} \equiv \sum \frac{(-1)^P}{(2n)!}$ permutations. If the set $\{C_k^{II'}\}$ is a complete set, then there is a one to one correspondance between the a- and ϱ-coefficients of (2.1) and (3.1).

By (3.3) a mapping between the fermionic state functionals (2.1) and bosonic state functionals (3.1) is established. The consequences of this mapping with respect to the functional equation (2.6) can be formulated by the following theorem:

Theorem 3.1. By the boson transformation (3.3) the functional fermion energy equation (2.6) is mapped onto the functional boson energy equation

$$
\begin{aligned}
\omega_{a0} P|\hat{\mathscr{B}}[b]\rangle &= 2R^{k}_{I_1M} K_{I_1I_2} b_k V\begin{pmatrix} I_2 M \\ k' \end{pmatrix} \partial_{k'} |\hat{\mathscr{B}}[b]\rangle - \sum_h W^{h}_{K_1I_2I_3I_4} \Bigg\{ 6F_{I_4M} R^{k}_{K_1M} b_k V\begin{pmatrix} I_2 I_3 \\ k' \end{pmatrix} \partial_{k'} \\
&\quad + 12 F_{I_4M} R^{k_1}_{K_1M'} R^{k_2}_{MM''} b_{k_1} b_{k_2} W\left(\begin{array}{c|c} I_2M' & I_3M'' \\ k_1' & k_2' \end{array}\right) \partial_{k_1'} \partial_{k_2'} - 2R^{k}_{K_1M} b_k W\left(\begin{array}{c|c} I_2I_3 & I_4M \\ k' & k'' \end{array}\right) \partial_{k'} \partial_{k''} \\
&\quad - \left(3F_{I_4K_2} F_{I_3K_3} + \tfrac{1}{4} A_{I_4K_2} A_{I_3K_3}\right) p \sum_{\lambda_1 \ldots \lambda_3} (-1)^P \frac{1}{3!} \\
&\quad \times \Bigg[12 R^{k}_{K_{\lambda_1}K_{\lambda_2}} R^{k'}_{K_{\lambda_3}M} b_k b_{k'} V\begin{pmatrix} I_2 M \\ k'' \end{pmatrix} \partial_{k''} + 8 R^{k_1}_{K_{\lambda_1}M_1} R^{k_2}_{K_{\lambda_2}M_2} R^{k_3}_{K_{\lambda_3}M_3} b_{k_1} b_{k_2} b_{k_3} W\left(\begin{array}{c|c} I_2M_3 & M_2M_1 \\ k_1' & k_2' \end{array}\right) \partial_{k_1'} \partial_{k_2'} \Bigg] \\
&\quad + \left(F_{I_4K_2} F_{I_3K_3} F_{I_2K_4} + \tfrac{1}{4} A_{I_4K_2} A_{I_3K_3} A_{I_2K_4}\right) p \sum_{\lambda_1 \ldots \lambda_4} (-1)^P \frac{1}{4!} \\
&\quad \times \Bigg[12 R^{k}_{K_{\lambda_1}K_{\lambda_2}} R^{k'}_{K_{\lambda_3}K_{\lambda_4}} b_k b_{k'} P + 48 R^{k}_{K_{\lambda_1}M} R^{k'}_{K_{\lambda_2}M'} R^{k''}_{K_{\lambda_3}K_{\lambda_4}} b_k b_{k'} b_{k''} V\begin{pmatrix} M'M \\ l \end{pmatrix} \partial_l \\
&\qquad + 16 R^{k_1}_{K_{\lambda_1}M_1} R^{k_2}_{K_{\lambda_2}M_2} R^{k_3}_{K_{\lambda_3}M_3} R^{k_4}_{K_{\lambda_4}M_4} b_{k_1} b_{k_2} b_{k_3} b_{k_4} W\left(\begin{array}{c|c} M_1M_2 & M_3M_4 \\ k_1' & k_2' \end{array}\right) \partial_{k_1'} \partial_{k_2'} \Bigg] \Bigg\} |\hat{\mathscr{B}}[b]\rangle
\end{aligned}
\tag{3.4}
$$

with the definitions

$$
P := \sum_{n=0}^{\infty} \frac{1}{n!} b_{k_1'} \ldots b_{k_n'} |0\rangle R^{k_1'}_{K_1K_2} \ldots R^{k_n'}_{K_{2n-1}K_{2n}} C^{\{K_1K_2}_{k_1} \ldots C^{K_{2n-1}K_{2n}\}}_{k_n} \langle 0| \partial_{k_n} \ldots \partial_{k_1}, \tag{3.5}
$$

$$
V\begin{pmatrix} I_1 I_2 \\ k \end{pmatrix} := \sum_{n=0}^{\infty} \frac{1}{n!} b_{k_1'} \ldots b_{k_n'} |0\rangle R^{k_1'}_{K_1K_2} \ldots R^{k_n'}_{K_{2n-1}K_{2n}} C^{\{I_1I_2}_{k} C^{K_1K_2}_{k_1} \ldots C^{K_{2n-1}K_{2n}\}}_{k_n} \langle 0| \partial_{k_n} \ldots \partial_{k_1}, \tag{3.6}
$$

$$
W\left(\begin{array}{c|c} I_1I_2 & I_3I_4 \\ k & k' \end{array}\right) := \sum_{n=0}^{\infty} \frac{1}{n!} b_{k_1'} \ldots b_{k_n'} |0\rangle R^{k_1'}_{K_1K_2} \ldots R^{k_n'}_{K_{2n-1}K_{2n}} C^{\{I_1I_2}_{k} C^{I_3I_4}_{k'} C^{K_1K_2}_{k_1} \ldots C^{K_{2n-1}K_{2n}\}}_{k_n} \langle 0| \partial_{k_n} \ldots \partial_{k_1},
$$

Proof. We write (2.6) in the general form

$$
\omega |\mathscr{F}[j]\rangle = \mathscr{H}[j, \partial] |\mathscr{F}[j]\rangle . \tag{3.8}
$$

By projection we obtain the corresponding equations for the matrix elements

$$
\omega \langle 0| \partial_{K_{2n}} \ldots \partial_{K_1} |\mathscr{F}[j]\rangle = \omega (-1)^n a^{K_{2n} \ldots K_1} = \langle 0| \partial_{K_{2n}} \ldots \partial_{K_1} \mathscr{H}[j, \partial] |\mathscr{F}[j]\rangle , \tag{3.9}
$$

and with (3.3) we obtain from (3.9) the equation

$$
\omega C^{\{K_1K_2}_{k_1'} \ldots C^{K_{2n-1}K_{2n}\}}_{k_n'} \varrho^{k_1' \ldots k_n'} = \langle 0| \partial_{K_{2n}} \ldots \partial_{K_1} \mathscr{H}[j, \partial] |\mathscr{F}[j]\rangle . \tag{3.10}
$$

Multiplication by $R^{k_1}_{K_1K_2} \ldots R^{k_n}_{K_{2n-1}K_{2n}}$ and summation yields with (3.5) the equivalent equation

$$
\omega \langle 0| \partial_{k_n} \ldots \partial_{k_1} P |\hat{\mathscr{B}}[b]\rangle = R^{k_1}_{K_1K_2} \ldots R^{k_n}_{K_{2n-1}K_{2n}} \langle 0| \partial_{K_{2n}} \ldots \partial_{K_1} \mathscr{H}[j, \partial] |\mathscr{F}[j]\rangle . \tag{3.11}
$$

To complete the proof, also the right-hand side of (3.11) has to be represented in terms of boson field matrix elements. In order to achieve this, each term of (2.6) has to be discussed separately. We first consider the term with $j\partial$. According to (3.11) and (2.6) we have

$$R^{k_1}_{K_1K_2}\dots R^{k_n}_{K_{2n-1}K_{2n}}\langle 0|\partial_{K_{2n}}\dots\partial_{K_1}K_{I_1I_2}j_{I_1}\partial_{I_2}|\mathscr{F}[j]\rangle =: T_1$$
$$=R^{k_1}_{K_1K_2}\dots R^{k_n}_{K_{2n-1}K_{2n}}K_{I_1I_2}\sum_{i=1}^{2n}(-1)^n a^{K_1\dots K_{i-1}I_2K_{i+1}\dots K_{2n}}\delta_{I_1K_i}$$
$$=R^{k_1}_{K_1K_2}\dots R^{k_n}_{K_{2n-1}K_{2n}}\sum_{i=1}^{2n}K_{I_1I_2}C^{\{K_1K_2\dots K_{i-1}I_2K_{i+1}\dots}_{k'_n}C^{K_{2n-1}K_{2n}\}}_{k'_n}\delta_{I_1K_1}\langle 0|\partial_{k'_n}\dots\partial_{k'_1}|\hat{\mathscr{B}}[b]\rangle . \tag{3.12}$$

Furthermore we evaluate

$$K_{I_1I_2}2R^k_{I_1M}\langle 0|\partial_{k_n}\dots\partial_{k_1}b_kV\begin{pmatrix}I_2M\\k'\end{pmatrix}\partial_{k'}|\hat{\mathscr{B}}[b]\rangle = T'_1$$
$$=K_{I_1I_2}2R^k_{I_1M}\sum_{i=1}^{n}\delta_{kk_i}\langle 0|\partial_{k_n}\dots\partial_{k_{i+1}}\partial_{k_{i-1}}\dots\partial_{k_1}V\begin{pmatrix}I_2M\\k'\end{pmatrix}\partial_{k'}|\hat{\mathscr{B}}[b]\rangle . \tag{3.13}$$

By means of (3.6) we obtain

$$T'_1=K_{I_1I_2}2R^k_{I_1M}\sum_{i=1}^{n}\delta_{kk_i}R^{k_1}_{K_1K_2}\dots R^{k_i}_{K_{2i-1}K_{2i}}\dots R^{k_n}_{K_{2n-1}K_{2n}}$$
$$\cdot C^{\{I_2M}_{k'}C^{K_1K_2}_{k'_1}\dots C^{K_{2i-1}K_{2i}}_{k'_i}\dots C^{K_{2n-1}K_{2n}\}}_{k'_n}\langle 0|\partial_{k'_n}\dots\partial_{k'_{i+1}}\partial_{k'_{i-1}}\dots\partial_{k'_1}\partial_{k'}|\hat{\mathscr{B}}[b]\rangle \tag{3.14}$$
$$=K_{I_1I_2}\sum_{i=1}^{n}R^{k_1}_{K_1K_2}\dots(R^{k_i}_{I_1M}-R^{k_i}_{MI_1})\dots R^{k_n}_{K_{2n-1}K_{2n}}C^{\{K_1K_2}_{k'_1}\dots C^{I_2M}_{k'_i}\dots C^{K_{2n-1}K_{2n}\}}_{k'_n}\langle 0|\partial_{k'_n}\dots\partial_{k'_1}|\hat{\mathscr{B}}[b]\rangle .$$

A comparison with (3.12) shows that $T_1=T'_1$. Thus we can rewrite (3.12) by

$$T_1=\langle 0|\partial_{k_n}\dots\partial_{k_1}\mathscr{H}_1[b,\partial]|\hat{\mathscr{B}}[b]\rangle \tag{3.15}$$

with

$$\mathscr{H}_1[b,\partial]:=2K_{I_1I_2}R^k_{I_1M}b_kV\begin{pmatrix}I_2M\\k'\end{pmatrix}\partial_{k'} . \tag{3.16}$$

The proof for the transformation of all other elements of $\mathscr{H}[j,\partial]$ runs along the same lines. So we do not explicitly discuss these terms. After having transformed all terms we can rewrite (3.11) in the form

$$\omega\langle 0|\partial_{k_n}\dots\partial_{k_1}P|\hat{\mathscr{B}}[b]\rangle$$
$$=\langle 0|\partial_{k_n}\dots\partial_{k_1}\hat{\mathscr{H}}[b,\partial]|\hat{\mathscr{B}}[b]\rangle . \tag{3.17}$$

As (3.17) holds for arbitrary projectors $\langle 0|\partial_{k_n}\dots\partial_{k_1}$, we obtain (3.4). □

With respect to the physical interpretation one has to compare the functional equation (3.4) with a "free" boson theory. By definition, free boson fields have to be abelian boson fields. In this case the general equation reads

$$\omega|\mathscr{B}\rangle_f=K_{kk'}b_k\partial_{k'}|\mathscr{B}\rangle_f+A_{kk'}b_kb_{k'}|\mathscr{B}\rangle_f , \tag{3.18}$$

and the boson field has a multiparticle free propagation with plane waves in $\mathbb{M}^4$. Any further term in a functional equation which is added to this free equation disturbs the free propagation and thus can be considered as a force acting on the boson field. Among such additional terms are those which can be reduced to the introduction of local field operator interactions in the original field equation of the bosons. These are terms which are consequences of the classical picture of pointlike bosons, thus they correspond to direct forces. On the other hand, (3.4) contains terms which reflect the nontrivial fermionic substructure of the bosons and thus have no classical counterpart on the pointlike boson level. Apart from formfactors which modify the classical interactions to nonlocal interactions, these additional terms are exchange terms resulting from the substructure of the bosons and in consequence lead to exchange forces between the bosons themselves. Just the latter terms are responsible for the complicated structure of (3.4). Thus, in treating (3.4) it is the first task to investigate the complications produced by exchange forces and to find ranges where these forces may be neglected.

4. Separation of Forces

The boson functional energy equation simultaneously contains the direct forces and the exchange forces. In order to study the effect of exchange forces we have to separate the exchange forces from the direct forces in (3.4). Obviously the principal part of exchange forces is contained in the terms (3.5), (3.6), (3.7). Thus we first treat these terms. We begin with (3.5).

Theorem 4.1. Let $\lambda := (\lambda_1 \dots \lambda_j)$ be a partition of n with

$$\lambda_i \in \mathbb{N}; \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_j; \quad \sum_{i=1}^{j} \lambda_i = n$$

and $s_1, \dots s_r$ the multiplicities of identical values of the $\lambda_1 \dots \lambda_j$, then the following relation holds:

$$R^{k'_1}_{K_1 K_2} \dots R^{k'_n}_{K_{2n-1} K_{2n}} C^{\{K_1 K_2}_{k_1} \dots C^{K_{2n-1} K_{2n}\}}_{k_n} \tag{4.1}$$

$$= \frac{n!}{(2n)!} 2^n \sum_{\text{part}\,\lambda} (-1)^j \frac{2^{\lambda_1 - 1} \dots 2^{\lambda_j - 1}\, n!}{s_1! \dots s_r!\, \lambda_1 \dots \lambda_j} \begin{Bmatrix} k'_1 \dots k'_n \\ k_1 \dots k_n \\ \lambda_1 \dots \lambda_j \end{Bmatrix}_{\text{sym}},$$

where

$$\begin{Bmatrix} k'_1 \dots k'_n \\ k_1 \dots k_n \\ \lambda_1 \dots \lambda_j \end{Bmatrix} := \{\mathrm{tr}[R^{k'_1} C_{k_1} \dots R^{k'_{\lambda_1}} C_{k_{\lambda_1}}] \dots \tag{4.2}$$

$$\dots \mathrm{tr}[R^{k'_{\lambda_{j-1}+1}} C_{k_{\lambda_{j-1}+1}} \dots R^{k'_{\lambda_j}} C_{k_{\lambda_j}}]\}$$

and

$$\{\,\}_{\text{sym}} := \frac{1}{n!} p \sum_{k_1 \dots k_n} \frac{1}{n!} \sum_{k'_1 \dots k'_n} \{\,\}. \tag{4.3}$$

Proof. Due to the permutations of $\{K_1 \dots K_{2n}\}$ the left-hand side of (4.1) is either symmetric with respect to $k_1 \dots k_n$ for fixed $k'_1 \dots k'_n$ or symmetric with respect to $k'_1 \dots k'_n$ for fixed $k_1 \dots k_n$. Any term on the left-hand side of (4.1) can be written as a partition of traces (4.2). If all terms of (4.1) are expressed in this way, we obtain

$$\text{l.h.s. } (4.1) = \frac{1}{(2n)!} \sum_{\text{part}\,\lambda_1 \dots \lambda_j} M_{\lambda_1 \dots \lambda_j} \begin{Bmatrix} k'_1 \dots k'_n \\ k_1 \dots k_n \\ \lambda_1 \dots \lambda_j \end{Bmatrix}, \tag{4.4}$$

where $\{M_{\lambda_1 \dots \lambda_j}\}$ are the corresponding multiplicity factors. For fixed $k_1 \dots k_n$ the r.h.s. of (4.4) must be symmetric in $k'_1 \dots k'_n$. Thus we have

$$\sum_{\text{part}\,\lambda} M_{\lambda_1 \dots \lambda_j} \begin{Bmatrix} k'_1 \dots k'_n \\ k_1 \dots k_n \\ \lambda_1 \dots \lambda_j \end{Bmatrix} \tag{4.5}$$

$$\equiv \sum_{\text{part}\,\lambda} p \sum_{k'_1 \dots k'_n} M_{\lambda_1 \dots \lambda_j} \frac{1}{n!} \begin{Bmatrix} k'_1 \dots k'_n \\ k_1 \dots k_n \\ \lambda_1 \dots \lambda_j \end{Bmatrix}.$$

On the other hand, the r.h.s. of (4.4) must be symmetric with respect to $k_1 \dots k_n$ for fixed $k'_1 \dots k'_n$. Thus we have with modified multiplicity factors $m_{\lambda_1 \dots \lambda_j}$

$$\sum_{\text{part}\,\lambda} M_{\lambda_1 \dots \lambda_j} \begin{Bmatrix} k'_1 \dots k'_n \\ k_1 \dots k_n \\ \lambda_1 \dots \lambda_j \end{Bmatrix}$$

$$= \sum_{\text{part}\,\lambda} m_{\lambda_1 \dots \lambda_j}\, p \sum_{k_1 \dots k_n} \begin{Bmatrix} k'_1 \dots k'_n \\ k_1 \dots k_n \\ \lambda_1 \dots \lambda_j \end{Bmatrix} \tag{4.6}$$

as an equivalent formulation of the symmetry property. With (4.5), (4.6) and (4.3) we then obtain

$$\text{l.h.s. } (4.1) = \frac{n!}{(2n)!} \sum_{\text{part}\,\lambda} m_{\lambda_1 \dots \lambda_j} \begin{Bmatrix} k'_1 \dots k'_n \\ k_1 \dots k_n \\ \lambda_1 \dots \lambda_j \end{Bmatrix}_{\text{sym}}. \tag{4.7}$$

After having explicitly symmetrized we can apply standard combinatorics by observing that the partitions can be realized by appropriate distributions of $k_1 \dots k_n$ values. First the simultaneous interchange of all C indices does not change the left-hand side of (4.1). Thus the multiplicity $m_{\lambda_1 \dots \lambda_j}$ must contain the factor 2^n. Furthermore, with respect to $k_1 \dots k_n$ the number of possible partitions leads to the factor $n!(s_1! \dots s_r!\, \lambda_1! \dots \lambda_j!)^{-1}$ if equal λ-values are taken into account. Finally the partitions allow internal rearrangements which yield the factor $2^{\lambda_1 - 1} \dots 2^{\lambda_j - 1} (\lambda_1 - 1)! \dots (\lambda_j - 1)!$, and the factor $(-1)^j$ comes from fermion permutations in (4.1). Thus we have

$$m_{\lambda_1 \dots \lambda_n} = \frac{2^{\lambda_1 - 1} \dots 2^{\lambda_j - 1}\, n! (\lambda_1 - 1)! \dots (\lambda_j - 1)!}{(s_1! \dots s_r!\, \lambda_1! \dots \lambda_j!)} 2^n (-)^j, \tag{4.8}$$

which yields the r.h.s. of (4.1). □

With respect to (3.6) the following theorem holds:

Theorem 4.2. Using the definitions of Theorem 4.1 the following relation holds:

$$R^{k'_1}_{K_1 K_2} \dots R^{k'_n}_{K_{2n-1} K_{2n}} C^{\{I_1 I_2}_{k_1} C^{K_1 K_2}_{k_2} \dots C^{K_{2n-1} K_{2n}\}}_{k_{n+1}} = \frac{1}{(2n+1)} R^{k'_1}_{K_1 K_2} \dots R^{k'_n}_{K_{2n-1} K_{2n}} C^{I_1 I_2}_{[k_1} C^{\{K_1 K_2}_{k_2} \dots C^{K_{2n-1} K_{2n}\}}_{k_{n+1}]} \tag{4.9}$$

$$+ \frac{(n+1)!\, 2^{n+1}}{(2(n+1))!} \sum_{\text{part}\,\lambda}^{n} (-1)^{j+1} \frac{2^{n-j+1}\, n!}{s_1! \dots s_r!\, \lambda_1 \dots \lambda_j} \begin{Bmatrix} k'_1 \dots k'_n\, I_1 I_2 \\ k_1 \dots k_n k_{n+1} \\ \lambda_1 \dots \lambda_j \end{Bmatrix}_{\text{sym}}$$

with $[n] = \sum \frac{1}{(n)!}$ permutation and

$$\begin{Bmatrix} k'_1 \dots k'_n \, I_1 I_2 \\ k_1 \dots k_n k_{n+1} \\ \lambda_1 \dots \lambda_j \end{Bmatrix} := \sum_{l=1}^{j} \lambda_l \operatorname{tr}[R^{k'_1} C_{k_1} \dots R^{k'_{\lambda_1}} C_{k_{\lambda_1}}] \dots [C_{k_{\lambda_l+1}} R^{k'_{\lambda_l+1}} \dots R^{k'_{\lambda_{l+1}-1}} C_{k_{\lambda+1}}]^{\{I_1 I_1\}} \dots$$
$$\dots \operatorname{tr}[R^{k'_{\lambda_{j-1}+1}} C_{k_{\lambda_{j-1}+2}} \dots R^{k'_{\lambda_j}} C_{k_{\lambda_j+1}}], \tag{4.10}$$

while the symmetrization (4.3) is generalized to $k_1 \dots k_{n+1}$ and $k'_1 \dots k'_n$.

Proof. Due to the permutations of $\{I_1 I_2 K_1 \dots K_{2n}\}$, the left-hand side of (4.9) is symmetric with respect to $k_1 \dots k_{n+1}$. Thus we have

$$\text{l.h.s. } (4.9) = R^{k'_1}_{K_1 K_2} \dots R^{k'_n}_{K_{2n-1} K_{2n}} C^{\{I_1 I_2}_{[k_1} C^{K_1 K_2}_{k_2} \dots C^{K_{2n-1} K_{2n}\}}_{k_{n+1}]}. \tag{4.11}$$

If we separate in $\{I_1 I_2 K_1 \dots K_{2n}\}$ those terms from the other terms where $I_1 I_2$ is exactly placed over one k_h, the symmetry in $[k_1 \dots k_{n+1}]$ can be used to replace $C^{I_1 I_2}_{kh}$ on the first position and we obtain the first term on the right-hand side of (4.9). In the remaining terms $I_1 I_2$ is distributed in such a way that two values k_α, k_β are involved. If the product on the left-hand side of (4.9) is factorized into a product of traces, this yields definition (4.10) (with omission of those terms already separated from (4.9)). Then the multiplicity factors have to be determined. Their calculations run along the same lines as those of Theorem 4.1. We thus do not discuss this in detail. □

The treatment of the last expression (3.7) is completely similar. Therefore, without further proof we formulate the corresponding theorem:

Theorem 4.3. Using the definitions of Theorem 4.1 and 4.2 the following relation holds:

$$R^{k'_1}_{K_1 K_2} \dots R^{k'_n}_{K_{2n-1} K_{2n}} C^{\{I_1 I_2}_{k_1} C^{I_3 I_4}_{k_2} C^{K_1 K_2}_{k_3} \dots C^{K_{2n-1} K_{2n}\}}_{k_{n+2}]}$$
$$= \frac{1}{(2n+3)(2n+1)} R^{k'_1}_{K_1 K_2} \dots R^{k'_n}_{K_{2n-1} K_{2n}} C^{\{I_1 I_2}_{[k_1} C^{I_3 I_4\}}_{k_2} C^{\{K_1 K_2}_{k_3} \dots C^{K_{2n-1} K_{2n}\}}_{k_{n+2}]}$$
$$+ \frac{(n+2)!\, 2^{n+2}}{(2n+4)!} \sum_{\text{part } \lambda}^{n} (-1)^{j+2} \frac{2^{n-j+2}\, n!}{s_1! \dots s_r!\, \lambda_1 \dots \lambda_j} \begin{Bmatrix} k'_1 \dots k'_n \, I_1 I_2 I_3 I_4 \\ k_1 \dots k_n k_{n+1} k_{n+2} \\ \lambda_1 \dots \lambda_j \end{Bmatrix}_{\text{sym}}, \tag{4.12}$$

where the bracket with $I_1 \dots I_4$ is the corresponding generalization of definition (4.10) and the symmetrization (4.3) is generalized to $k_1 \dots k_{n+2}$ and $k'_1 \dots k'_n$. □

5. Estimate of Exchange Forces

After having evaluated those terms of (3.4) which correspond to exchange forces, we have to investigate the problem under which conditions exchange forces are essential for composite boson physics or under which conditions they can be neglected. For a first draft we will not use complicated (i.e. realistic) boson functions $\{C^{I_1 I_2}_k\}$. Rather, with respect to exchange forces we assume that the essential property of wave functions $\{C^{I_1 I_2}_k\}$ is their spatial extension. In discussing only this aspect we restrict ourselves to a simple class of wave functions which are used in nuclear physics, namely oscillator functions. We thus consider the set

$$\{C^{I_1 I_2}_k\} = \{R^{k}_{I_1 I_2}{}^{\times}\} := \left\{ \frac{m^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}} \pi^{\frac{3}{4}}} \exp\left(i\boldsymbol{k}\boldsymbol{z} - \frac{m^2}{2} \boldsymbol{u}^2 \right),\ \boldsymbol{k} \in \mathbb{R}^3 \right\} \tag{5.1}$$

with $\boldsymbol{z} = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2)$, $\boldsymbol{u} = (\boldsymbol{r}_1 - \boldsymbol{r}_2)$ and $m = \sigma^{-1}$ for $\hbar = c = 1$, where σ is the length scale expressed by the mass scale m.

The wave functions (5.1) describe moving boson states with momentum k relative to the general frame of reference of the functional equation (3.4). The scale factor $m=\sigma^{-1}$ in (5.1) defines the spatial extension of this function. According to special relativity, length scales σ of moving objects are contracted with respect to observers in a rest frame. Thus m cannot be a constant but has to be transformed inversely to σ, i.e., $m=m(\boldsymbol{k})$. To calculate $m=m(\boldsymbol{k})$ we assume that the composite bosons have a rest mass μ. Then $k^2=\mu^2$, i.e. $k_0^2=\mu^2+\boldsymbol{k}^2$. Furthermore it is $k_0^2=E^2=\mu^2(1-\beta^2)^{-1}$ with $\beta^2=v^2$ for $\hbar=c=1$, and therefore by comparison $(1-\beta^2)^{-1}=(1+\boldsymbol{k}^2/\mu^2)$. From this it follows

$$m(\boldsymbol{k}) = m(1+\boldsymbol{k}^2/\mu^2)^{\frac{1}{2}},$$

which has to replace m in (5.1).

With the set (5.1) for any m any trace expression can exactly be calculated. However, further evaluation of the exchange terms can considerably be simplified if we consider large m. In this case we have

$$\lim_{m\to\infty} m^3(2\pi)^{-\frac{3}{2}} \exp\left(-\frac{m^2}{2}\boldsymbol{u}^2\right) = \delta(\boldsymbol{u}),$$

and thus we can approximately replace the oscillator functions by δ-distributions. This yields

$$\operatorname{tr}[R^{k'_1} C_{k_1}\dots R^{k'_l} C_{k_l}] = \delta\left(\sum_{i=1}^{l}(\boldsymbol{k}_i-\boldsymbol{k}'_i)\right)(\pi^{-\frac{3}{2}})^{l-1}\prod_{i=1}^{l} m^{-\frac{3}{2}}(\boldsymbol{k}_i)\, m^{-\frac{3}{2}}(\boldsymbol{k}'_i)\, m^3(\boldsymbol{k}_l) \tag{5.2}$$

and

$$\operatorname{tr}[C_k R^{k'_1} C_{k_1}\dots R^{k'_l} C_{k_l}]^{I_1 I_2} \approx C_k^{I_1 I_2} \exp\left[iz\left(\sum_{i=1}(\boldsymbol{k}_i-\boldsymbol{k}'_i)\right)\right]\prod_{i=1} m^{-\frac{3}{2}}(\boldsymbol{k}_i)\, m^{-\frac{3}{2}}(\boldsymbol{k}'_i)(\pi^{-\frac{3}{2}})^l. \tag{5.3}$$

In the appendix the following inequalities are proven:

$$\left|\int\{\operatorname{tr}[R^{k'_1} C_{k_1}\dots R^{k'_{\lambda_1}} C_{k_{\lambda_1}}]\dots\operatorname{tr}[R^{k'_{n-\lambda_j+1}} C_{k_{n-\lambda_j+1}}\dots R^{k'_n} C_{k_n}]\}_{\text{sym}}\,\hat{\varrho}^{k_1\dots k_n}\,dk_1\dots dk_n\right| \le \left(a\frac{\mu}{m}\right)^{3(n-j)} |\hat{\varrho}|_{\max} \tag{5.4}$$

and

$$\Big|\int\{\operatorname{tr}[R^{k'_1} C_{k_1}\dots R^{k'_{\lambda_1}} C_{k_{\lambda_1}}]\dots[C_k R^{k'_{s+1}} C_{k_{s+1}}\dots R^{k'_{s+\lambda_l}} C_{k_{s+\lambda_l}}]^{\{I_1 I_2\}_{\text{as}}}$$
$$\dots\operatorname{tr}[R^{k'_{n-\lambda_j+1}} C_{k_{n-\lambda_j+1}}\dots R^{k'_n} C_{k_n}]\}_{\text{sym}}\,\hat{\varrho}^{k_1\dots k_n}\,dk_1\dots dk_n\Big| \le |C_k^{I_1 I_2}|\left(a\frac{\mu}{m}\right)^{3(n-j+1)} |\hat{\varrho}|_{\max}, \tag{5.5}$$

where $s := \sum_{i=1}^{l-1}\lambda_i$.

By means of these inequalities for (4.1) an estimate can be given. We decompose (4.1) into the partition $\lambda_i=1$, $i=1\dots n$ and the rest. This yields

$$R^{k'_1}_{K_1K_2}\dots R^{k'_n}_{K_{2n-1}K_{2n}}\, C^{\{K_1K_2}_{k_1}\dots C^{K_{2n-1}K_{2n}\}}_{k_n} = \frac{n!}{(2n!)}\,2^n\left(p\sum_{\lambda_1\dots\lambda_n}\frac{1}{n!}\,\delta_{k'_1 k_{\lambda_1}}\dots\delta_{k'_n k_{\lambda_n}} + R_1\right). \tag{5.6}$$

If we consider the left-hand side of (5.6) as an integral kernel which acts on $\hat{\varrho}$, we obtain by means of (5.4)

$$|R_1| \le \sum_{j=1}^{n-1}\sum_{\text{part}\,\lambda}\frac{2^{n-j}\,n!}{s_1!\dots s_r!\,\lambda_1\dots\lambda_j}\left(a\frac{\mu}{m}\right)^{3(n-j)}. \tag{5.7}$$

For a further estimate of (5.7) we apply the following inequalities

$$\frac{n!}{s_1!\dots s_r!\,\lambda_1!\dots\lambda_j!} \le (n^2)^{n-j} \tag{5.8}$$

and

$$2^{n-j}(\lambda_1-1)!\dots(\lambda_j-1)! < (2n)^{n-j} \tag{5.9}$$

without proof. This gives

$$|R_1| \le \sum_{j=1}^{n-1}\Delta_j(n)\left(\frac{an\mu}{m}\right)^{3(n-j)}, \tag{5.10}$$

where $\Delta_j(n)$ denotes the number of partitions with exactly j trace factors in (4.1). If $P_j(n)$ is defined by

$$P_j(n) := \sum_{\alpha=1}^{j}\Delta_\alpha(n), \tag{5.11}$$

then the relation

$$\Delta_j(n) = P_j(n-j) \tag{5.12}$$

holds. Thus we can rewrite (5.10) as

$$|R_1| \leq \sum_{k=1}^{n-1} P(k)\left(\frac{a\,n\,\mu}{m}\right)^{3k}, \tag{5.13}$$

and by means of the generating function for the $P(k)$, for $(a\,n\,\mu/m)^3 \ll 1$ we get the inequality

$$|R_1| \leq n^3\left(\frac{a\,\mu}{m}\right)^3. \tag{5.14}$$

In an analogous way (4.10) can be estimated. This expression can be written in the form

$$R^{k'_1}_{K_1K_2} \dots R^{k'_n}_{K_{2n-1}K_{2n}}\, C^{\{I_1I_2}_{k_1}\, C^{K_1K_2}_{k_2} \dots C^{K_{2n-1}K_{2n}\}}_{k_{n+1}} = \frac{1}{(2n+1)}\, R^{k'_1}_{K_1K_2} \dots R^{k'_n}_{K_{2n-1}K_{2n}}\, C^{I_1I_2}_{[k_1}\, C^{\{K_1K_2}_{k_2} \dots C^{K_{2n-1}K_{2n}\}}_{k_{n+1}]} + R_2 \tag{5.15}$$

with

$$\frac{1}{(2n+1)}\, R^{k'_1}_{K_1K_2} \dots R^{k'_n}_{K_{2n-1}K_{2n}}\, C^{I_1I_2}_{[k_1}\, C^{\{K_1K_2}_{k_2} \dots C^{K_{2n-1}K_{2n}\}}_{k_{n+1}]}$$
$$= \frac{(2n)!\,2(n+1)}{(2(n+1))!}\,\frac{2^n\,n!}{(2n)!} \sum_{\lambda_1\dots\lambda_{n+1}} \frac{1}{(n+1)!}\, C^{I_1I_2}_{k_{\lambda_1}}\,\delta_{k_{\lambda_2}k'_1} \dots \delta_{k_{\lambda_{n+1}}k'_n} \tag{5.16}$$

and

$$|R_2| \leq \frac{(n+1)!\,2^{n+1}}{(2(n+1))!}\,|C^{I_1I_2}_k|\,n\left(\frac{a\,\mu}{m}\right)^3. \tag{5.17}$$

Finally we obtain for the first term on the right-hand side of (4.12)

$$\frac{1}{(2n+3)(2n+1)}\, R^{k'_1}_{K_1K_2} \dots R^{k'_n}_{K_{2n-1}K_{2n}}\, C^{I_1I_2}_{[k_1}\, C^{I_3I_4}_{k_2}\, C^{\{K_1K_2}_{k_3} \dots C^{K_{2n-1}K_{2n}\}}_{k_{n+2}]}$$
$$= \frac{(n+2)!}{(2(n+2))!}\,2^{n+2}\,\frac{1}{8} \sum_{\substack{\lambda_1\dots\lambda_4\\ \lambda'_1\dots\lambda'_{n+2}}} (-1)^P\,\frac{1}{(n+2)!}\, C^{I_{\lambda_1}I_{\lambda_2}}_{k_{\lambda'_1}}\, C^{I_{\lambda_3}I_{\lambda_4}}_{k_{\lambda'_2}}\,\delta_{k'_1k_{\lambda_3}} \dots \delta_{k'_nk_{\lambda_{n+2}}}. \tag{5.18}$$

For brevity we do not cite the inequality of the rest on the right-hand side of (4.12).

If we define a new state functional by

$$|\mathscr{B}[b]\rangle := \sum_{n=0}^{\infty} \frac{1}{n!}\,\frac{2^n\,n!}{(2n)!}\,\hat{\varrho}^{k_1\dots k_n}\, b_{k_1}\dots b_{k_n}|0\rangle =: \sum_{n=0}^{\infty} \frac{1}{n!}\,\varrho^{k_1\dots k_n}\, b_{k_1}\dots b_{k_n}|0\rangle \tag{5.19}$$

from (5.6), (5.16) and (5.18) we obtain the equations

$$\langle 0|\,\partial_{k_n}\dots\partial_{k_1} P|\hat{\mathscr{B}}[b]\rangle = \langle 0|\,\partial_{k_n}\dots\partial_{k_1}|\mathscr{B}[b]\rangle, \tag{5.20}$$

$$\langle 0|\,\partial_{k_n}\dots\partial_{k_1} V\begin{pmatrix} I_1I_2\\ k\end{pmatrix}\partial_k|\hat{\mathscr{B}}[b]\rangle = \langle 0|\,\partial_{k_n}\dots\partial_{k_1}\, C^{I_1I_2}_k\,\partial_k|\mathscr{B}[b]\rangle, \tag{5.21}$$

$$\langle 0|\,\partial_{k_n}\dots\partial_{k_1} W\left(\begin{matrix} I_1I_2\\ k\end{matrix}\middle|\begin{matrix} I_3I_4\\ k'\end{matrix}\right)\partial_k\partial_{k'}|\hat{\mathscr{B}}[b]\rangle = \langle 0|\,\partial_{k_n}\dots\partial_{k_1}\, 3\,C^{\{I_1I_2}_k\, C^{I_3I_4\}}_{k'}\,\partial_k\partial_{k'}|\mathscr{B}[b]\rangle \tag{5.22}$$

if R_1, R_2 and R_3 are neglected.

If (5.20), (5.21), (5.22) are substituted in (3.4), the following boson functional equation results:

$$\begin{aligned}\omega_{a0}|\mathscr{B}[b]\rangle = {} & 2R^k_{I_1M}K_{I_1I_2}b_k C^{I_2M}_{k}\partial_{k'}|\mathscr{B}[b]\rangle - \sum_h W^h_{K_1\{I_2I_3I_4\}}\Big\{6F_{I_4M}R^k_{K_1M}b_k C^{I_2I_3}_{k'}\partial_{k'} \\ & +36F_{I_4M}R^{k_1}_{K_1M'}R^{k_2}_{MM''}b_{k_1}b_{k_2}C^{\{I_2M'}_{k'_1}C^{I_3M''\}}_{k'_2}\partial_{k'_1}\partial_{k'_2} - 6R^k_{K_1M}b_k C^{\{I_2I_3}_{k'}C^{I_4M\}}_{k''}\partial_{k'}\partial_{k''} \\ & -(3F_{I_4K_2}F_{I_3K_3}+\tfrac{1}{4}A_{I_4K_2}A_{I_3K_3}) \\ & \cdot p\sum_{\lambda_1\ldots\lambda_3}(-1)^P\frac{1}{3!}[12R^k_{K_{\lambda_1}K_{\lambda_2}}R^{k'}_{K_{\lambda_3}M}b_k b_{k'}C^{I_2M}_{k''}\partial_{k''}+24R^{k_1}_{K_{\lambda_1}M_1}R^{k_2}_{K_{\lambda_2}M_2}R^{k_3}_{K_{\lambda_3}M_3}b_{k_1}b_{k_2}b_{k_3}C^{\{I_2M_3}_{k'_1}C^{M_2M_1\}}_{k'_2}\partial_{k'_1}\partial_{k'_2}] \\ & +(F_{I_4K_2}F_{I_3K_3}F_{I_2K_4}+\tfrac{1}{4}A_{I_4K_2}A_{I_3K_3}A_{I_2K_4}) \\ & \cdot p\sum_{\lambda_1\ldots\lambda_4}(-1)^P\frac{1}{4!}[12R^k_{K_{\lambda_1}K_{\lambda_2}}R^{k'}_{K_{\lambda_3}K_{\lambda_4}}b_k b_{k'}+48R^k_{K_{\lambda_1}M}R^{k'}_{K_{\lambda_2}M'}R^{k''}_{K_{\lambda_3}K_{\lambda_4}}b_k b_{k'}b_{k''}C^{M'M}_{l}\partial_l \\ & +48R^{k_1}_{K_{\lambda_1}M_1}R^{k_2}_{K_{\lambda_2}M_2}R^{k_3}_{K_{\lambda_3}M_3}R^{k_4}_{K_{\lambda_4}M_4}b_{k_1}b_{k_2}b_{k_3}b_{k_4}C^{\{M_1M_2}_{k'_1}C^{M_3M_4\}}_{k'_2}\partial_{k'_1}\partial_{k'_2}]\Big\}|\mathscr{B}[b]\rangle\,. \end{aligned} \tag{5.23}$$

Within the (μ/m)-approximation, (5.23) can be further simplified. If (5.23) is projected in configuration space, terms which differ by the factor $b_k\partial_l$ act on the same amplitude $\varrho^{k_1\ldots k_n}\ \forall k_1\ldots k_n$, $n=1\ldots\infty$. Thus between such terms an estimate can be made wih respect to their relative magnitude. Using (5.3), it follows that the term with $b_{k_1}b_{k_2}b_{k_3}\partial_{k'_1}\partial_{k'_2}$ is proportional to $(\mu/m)^3 b_k b_{k'}\partial_{k''}$. Thus, with respect to the term $b_k b_{k'}\partial_{k''}$ the term $b_{k_1}b_{k_2}b_{k_3}\partial_{k'_1}\partial_{k'_2}$ can be neglected. Analogous relations hold for the terms with $b_{k_1}b_{k_2}b_{k_3}b_{k_4}\partial_{k'_1}\partial_{k'_2}$, $b_k b_{k'}b_{k''}\partial_l$ and $b_k b_{k'}$. Of the latter terms only the term $b_k b_{k'}$ is left within this approximation.

Taking into account these estimates, from (5.23) we obtain the equation

$$\begin{aligned}\omega_{a0}|\mathscr{B}[b]\rangle = {} & 2R^k_{I_2M}K_{I_1I_2}C^{I_2M}_{k}b_k\partial_{k'}|\mathscr{B}[b]\rangle - \sum_h W^h_{K_1\{I_2I_3I_4\}}\Big\{6F_{I_4M}R^k_{K_1M}C^{I_2I_3}_{k'}b_k\partial_{k'} \\ & +36F_{I_4M}R^{k_1}_{K_1M'}R^{k_2}_{MM''}C^{\{I_2M'}_{k'_1}C^{I_3M''\}}_{k'_2}b_{k_1}b_{k_2}\partial_{k'_1}\partial_{k'_2} - 6R^k_{K_1M}C^{\{I_2I_3}_{k'}C^{I_4M\}}_{k''}b_k\partial_{k'}\partial_{k''} \\ & -(3F_{I_4K_2}F_{I_3K_3}+\tfrac{1}{4}A_{I_4K_2}A_{I_3K_3})\,12R^k_{\{K_1K_2}R^{k'}_{K_3\}M}C^{I_2M}_{k''}b_k b_{k'}\partial_{k''} \\ & +(F_{I_4K_2}F_{I_3K_3}F_{I_2K_4}+\tfrac{1}{4}A_{I_4K_2}A_{I_3K_3}F_{I_2K_4})\,12R^k_{\{K_1K_2}R^{k'}_{K_3K_4\}}b_k b_{k'}\Big\}|\mathscr{B}[b]\rangle\,. \end{aligned} \tag{5.24}$$

Equation (5.24) was first calculated in [2]. The derivation of (5.24) in [2] is based on a technique which was developed by one of the authors. By means of a special functional chain rule with subsequent omission of those terms which are neglected within the (μ/m)-approximation one can derive (5.24) by a shortcut calculation. By the theorems given in this paper this technique is justified provided exchange forces can be neglected.

Appendix

Due to the formfactors which act in the high energy range of the boson equation (3.4), this equation corresponds to a highly regularized field theory. We thus assume that the boson amplitudes $\hat{\varrho}^{k_1\ldots k_n}$ are bounded $\forall \boldsymbol{k}_1\ldots\boldsymbol{k}_n$, $n=1\ldots\infty$. If $|\hat{\varrho}|_{\max}$ is the amount of the maximum value of $\hat{\varrho}^{k_1\ldots k_n}$ and if (5.2) is taken into account, we have the estimate

$$\text{l.h.s. (5.4)} \le |\hat{\varrho}|_{\max}\prod_{i=1}^{j}A_i \tag{A.1}$$

with

$$A_i := (\pi^{-\frac{3}{2}})^{\lambda_i-1}\prod_{h=1}^{\lambda_i}m(\boldsymbol{k}'_{h+s_{i-1}})^{-\frac{3}{2}}\,m(\boldsymbol{k}'_{s_{i-1}})^3 \tag{A.2}$$

$$\cdot\int\prod_{h=1}^{\lambda_i}m(\boldsymbol{k}_h)^{-\frac{3}{2}}\,\delta\left(\sum_{h=1}^{\lambda_i}(\boldsymbol{k}_h-\boldsymbol{k}'_{h+s_{i-1}})\right)\mathrm{d}^3k_1\ldots\mathrm{d}^3k_{\lambda_i}$$

with $s_i := \sum_{k=1}^{i} \lambda_k$. The integral in (A.2) can be transformed into the expression

$$A_i \leq (\pi^{-\frac{3}{2}})^{\lambda_i - 1} \prod_{h=1}^{\lambda_i} m(\boldsymbol{k}'_{h+s_{i-1}})^{-\frac{3}{2}} m(\boldsymbol{k}'_{s_{i-1}})^3 \frac{(\mu^3)^{\lambda_i - 1}}{(m^{\frac{3}{2}})^{\lambda_i}}$$

$$\cdot \int \prod_{i=1}^{\lambda_i - 1} (1+\boldsymbol{x}_i^2)^{-\frac{3}{4}} \, \mathrm{d}^3 x_1 \dots \mathrm{d}^3 x_{\lambda_i - 1} \,. \qquad (A.3)$$

Observing that $m(0) \leq m(k)$ we obtain from (A.3)

$$A_i \leq (\pi^{-\frac{3}{2}})^{\lambda_i - 1} \left(\frac{\mu^3}{m^3} \right)^{\lambda_i - 1}$$

$$\cdot \int \prod_{i=1}^{\lambda_i - 1} (1+\boldsymbol{x}_i^2)^{-\frac{3}{4}} \, \mathrm{d}^3 x_1 \dots \mathrm{d}^3 x_{\lambda_i - 1} \,. \qquad (A.4)$$

Due to our substitution of oscillator functions by δ-distributions, in the Riemannian sense the integral in (A.4) is slightly divergent. Nevertheless, according to [5] this integral can be made finite in distribution theory. Without going into details we thus finally obtain

$$\text{l.h.s. } (5.4) \leq |\hat{\varrho}|_{\max} \left(a \frac{\mu^3}{m^3} \right)^{n-j}, \qquad (A.5)$$

where $a \approx 4$. An analogous estimate holds for (5.5).

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